Accuracy in Recursive Minimal State Space Methods

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Accuracy in Recursive Minimal State Space Methods *

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Abstract

We identify a critical condition, based on some qualitative properties of the expected marginal utility of consumption, that insure the accurate performance of frequently used methods in recursive macroeconomics. This condition can be found in a large fraction of applied papers. Moreover, in a model which does not satisfy the mentioned condition, we measure the bias of solutions using a closed form continuous recursive equilibrium. We found 2 sources of inaccuracy in minimal state space methods: the lack of a convergent operator and the inexistence of a well defined (stochastic) steady state. We found that a canonical procedure may sub-estimate (over-estimate) concentration (dispersion) measures with respect to the ergodic distribution of the model. It is shown that even a numerically convergent minimal state space (MSS) algorithm may not match the ergodic distribution of the model as the MSS equilibrium might not have a well defined steady state. These facts imply in turn that the computed effects of economic policies are also inacurate. Moreover, we identify a connection between the lack of convergence in the MSS algorithm and the equilibrium budget constraint which implies that simulated paths are distorted in any time period.

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1 Introduction

Sometimes macroeconomics is about new answers to old questions. This is the take away point of [14]. The authors analysis is based on the lack of exogenous variability in the field. Even though [14] used a fairly broad approach, they does not exhaust the reasons behind the state of the art in the literature. Even if we were able to find an exogenous policy shock, in case we would like to perform a structural analysis, models may not have closed form solutions. Thus, numerical methods are a fundamental ingridient in any macroeconomic model.

Since the seminal paper of [13] macroeconomists have been used the recursive representation of sequential equilibria to solve and simulate models. There are numerical and theoretical reasons behind this choice. As regards the former, it is easier to numerically approximate a first order stationary dynamic process rather than the sequential representation originally defined. In reference to the latter, a markovian structure allows to define a well behaved long term equilibria (i.e. a steady state) using a recursive equilibrium notion (see for instance [6]). Finally, and more importantly, the theoretical and computational arguments are related with each other since accurate numerical simulations requeres a Markovian representation and an appropriate steady state (see for intance [20] among others).

This paper has 2 contributions. First, we identify a critical condition, based on the some qualitative properties of the expected marginal utility of consumption, that insure the accurate performance of frequently used methods in recursive macroeconomics. This condition can be found in a large fraction of applied papers. Second, we measure the bias of solutions in a model which does not satisfy the mentioned condition using a closed form continuous recursive equilibrium. We found 2 sources of inaccuracy: the lack of a convergent operator and the inexistence of a well defined (stochastic) steady state. For expositional purposes, our findings are based on a canonical RBC model. However, the analysis can be extended to other branches of the literature as the identified condition is present in several macro models.

More precisely, the purpose of this paper is to show that, in certain environments, it is possible to obtain a recursive representation with an ergodic invariant measure, a finite number of exogenous shocks and a well behaved state space (i.e. compact). One of the main contributions of the paper is to present a closed form continuous Markov equilibrium that satisfies all the requirements of the sequential version of a canonical RBC model with taxes. Equipped with that equilibrium it is possible to test the accuracy of simulations of a standard, even numerically convergent, minimal state space algorithm. *We found that a cononical procedure may sub-estimate (over-estimate) concentration (dispersion) measures with respect to the ergodic distribution of the model.*

When it comes to compute recursive equilibrium models, the curse of dimentionality calls for minimal state space (MSS) methods. However, [12] argued that in the presence of multiple equilibria a MSS recursive representation may not exist. As uniqueness has been an elusive quest in this field[1], this fact justifies the necessity of an enlarged state space in an incomplete markets general equilibrium framework. By enlarging the number of variables in the state space, we show that it is possible to obtain *multiple* markovian representations, one of them *continuous with a stationary state space*. These last "selection" allows us to derive a well defined steady state by applying standard results.

We test the accuracy of simulations in MSS recursive equilibrium methods using a *closed form* generalized markovian equilibrium (GME) for a standard version of the RBC model with decreacing taxes on capital presented in [19]. As all MSS recursive equilibria form a subset of all GME, if both equilibrium types are well defined in the long run, any simulation from the latter must be matched using the former. It is shown that even a numerically convergent MSS algorithm *may not match the ergodic distribution of the model* as the MSS equilibrium might not have a well defined steady state. The bias not only affects the long run simulations derived from MSS methods but also the trajectories obtained from them as state of the art algorithms may not converge. These facts imply in turn that the computed effects of economic policies are also innacurate. Moreover, we identify a connection between the lack of convergence in the MSS algorithm and the equilibrium budget constraint which implies that simulated paths are distorted in any time period.

The paper is organized as follows: section 2 present an overview og the main results using a non-stochastic simple economy. Section 3 presents the canonical model and the closed form recursive equilibrium and discusses its implications. Section 4 presents the numerical test. Section 5 concludes.

1.1 Relation with the literature

Most, if not any, "macro" model includes an Euler equation. In particular, the expected marginal utility of consumption satisfies:

$$\beta E_z \left[u'(c_+(K)(1-\tau(K))R(K) \right] \tag{1}$$

Where c_+ denotes consumption "tomorrow", Z is a exogenous shock, τ is a tax rate on assets K and R is the gross rate of return. In a model of with production (as in [19]), K denotes capital and R its marginal product. In a small open economy model, R is exogenous, τ is a tax on debt ($-K$) and represent a macro-prudential policy (see [5]). In the default literature, $\tau = 0$ and R depends on the probability of default, which in turn is a function of debt (see [4]).

[1] [8] provided conditions to guarantee the uniqueness of equilibria in an infinite horizon economy with complete markets. There is no analogous result for incomplete markets

Following [1] and [7], in this paper we show that if equation (1) is *monotonic* in K it is possible to derive an accurate algorithm that converges to a well defined recursive equilibrium. For instance, in the small open economy literature, as τ is decreasing in K but consumption is tipically increasing, equation (1) is not monotonic. In the deafult literature, R is decreasing in K and thus (1) is decreasing, so the results in [7] insure the existence of a computable and numerically efficient algorithm. In the RBC literature, if τ is decreasing in K, (1) is not monotonic. The purpose of this paper is to measure the accuracy of state of the art algorithms when the monotonicity of (1) is not satisfied.

Contrarily to what is done in the numerical literature (see for instance [2]), we can measure this bias using an accurate closed form solution which also has a well behaved stady state. Thus, we can measure the short and long run implications of missing the monotonicity of (1). [20] have perform a similar excercise without an accurate closed form solution for optimal economies. We extend those results for models with distortions.

From a theoretical point of view, we sharpen the characterization of ergodic recursive equilibrium in [6]. We provide a counterexample for the equivalence between a continuous markovian representation and the uniqueness of the sequential equilibrium. In words of [6]:

"the existence of a continuos selection - tantamount to the uniqueness of equilibrium in each state - is not often satisfied".

We found a stationary (i.e. time independent) recursive representation with multiple equilibrium in some nodes which has a continuous selection. This result is relevant to relax recently found conditions to insure the existence of a ergodic steady state. These conditions are at odds with the computation of the model as they involve a large number of continuations for each node (see [21]). The existence of a continuous selection in a model with a finite number of shocks is essential to insure the ergodicity of simulations in a computable framework.

2 Preview of the results in a deterministic economy

Imagine a cononical RBC model distorted by ad-valorem taxes. As a distinctive fact, the aliquot is allowed to vary along with the business cycle. In particular, it will be assumed that it is *decreasing* in the aggregate state of the economy. There is an infinitely lived representative agent endowed with k_0 units of capital. She must choose a sequence of consumption and savings for each unit of time, denoted $t \geqslant 0$, in order to maximize her lifetime utility. For simplicity, we will assume for now that there is no uncertainty. Capital deprecites enterily after 1 period. Accumulated saving are rented to a firm, which is assumed to maximize profits using a decreasing returns to scale technology represented by a strongly concane production function. There is a Government that levies an ad-valorem tax on rental income. As mentioned, the aliquot depends on the aggregate state of the economy, denoted K, even though this connection is not perceived by the agent.

The Government rebates back the collected taxes making lump-sum transfers to the agent. Finally, as the agent owns the capital stock, she receives the profits form the firm.

Thus, the flow budget constraint of the agent is:

$$c_t + x_t = \pi_t(K_t) + (1 - \tau(K_t))r(K_t)k_t + T_t(K_t)$$

Where τ is the aliquot for the ad-valorem tax, r is the rental rate, π denotes profits, x_t represents investment and, due to full depreciation, k_{t+1} and T_t are transfers. The Goverment runs a balanced budget, $\tau(K_t)r(K_t)k_t = T_t$, and profit maximization implies $F(K_t) = \pi_t(K_t) + r(K_t)K_t$ where $r(K_t) = F'(K_t)$ and $F(K_t)$ denotes aggregate output. As there is a single firm, the only price in the economy, r, is a function of aggregate capital, K. Moreover, tax collection is allowed to explicitly depend on the aggregate state of the economy in order to capture the interaction between the business cycle and fiscal policy.

Replacing the equilibrium conditions in the flow budget constraint, we get:

$$c_t + x_t = F(K_t) + (k_t - K_t)F'(K_t)$$

The above equation is the aggregate budget constraint. Note that gross rental income is proportional to individual capital holdings, k. In equilibruim, it will be required that $k = K$. Thus, *if we can insure that individual and aggregate capital stocks remains closed to each other along the computed equilibrium trajectories, we will say that the decentralized equilibrium is not distorted*. Now suppose we want to solve the model and simulate this economy. The canonical approach since [17] is to use the associated dynamic programming program and the policy functions derived from it. In this framework, the agent is supposed to solve:

$$V(k, K) = Max_{c,x}\ u(c) + \beta V(k', K')$$

Subject to

$$x, c \in [0, \pi_t(K_t) + (1 - \tau(K_t))r(K_t)k_t + T_t(K_t)]$$

$$x + c = \pi_t(K_t) + (1 - \tau(K_t))r(K_t)k_t + T_t(K_t)$$

$$K' = G(K)$$

Where G is the *perceived* law of motion for aggregate capital. A *minimal state space recursive equilibrium* (MSSRE) in this economy is a pair of policy functions $c(k,K), x(k,K)$ such that:

$$c(k,K) + x(k,K) = F(K)$$

$$\tau(K)r(K)k = T$$

$$x(k,K) = G(K)$$

$$k = K$$

$$r(K) = F'(K)$$

We call $x(k,K) = G(K)$ the *rational expectations condition.* Now suppose we want to simulate the economy, given $k_0 = K_0$. We can use the set of policy functions iteratively. In order to take care of rational expectations condition, note that the Bellman equation above define a mapping $T(G_n)(k,K) \longmapsto G_{n+1}(k,K)$, where $T(G_n)(k,K) = x(k,K;G_n)$. In a recursive equilibrium the perceived law of motion G_* satisfies $T(G_*)(K,K) = x(K,K;G_*) = G_*(K)$.

Since [17] it is frequent to iterate on T, starting from an arbitrary initial condition, obtaining a sequence $\{G_n\}_n$. *Will G_n converge to G_*?* Equivalently, is there a numerically implementable operator that converge to a recursive equlibrium? Since [1], we know that if $(1-\tau(K))F'(K)$ is *decreasing* (in K), we can provide a positive answer to this question. Unfortunately, there are some cases where this condition does not hold for any K. In this paper, as in [19], we assume that τ is *decreasing* in K, which in turn implies that $(1-\tau(K))F'(K)$ is not monotonic as F is strictly concave.

More to the point, if we iterate on T, the limiting function G_∞ satisfies:

$$c^{'} + x^{'} = F(K') + (x(K,K;G_\infty) - K')F'(K')$$

Where $K' = G_\infty(K)$. Now, if the numerical procedure does not converge, we know that the perceived G and the actual x law of motion for capital will not be equal, at least for some K. That is, $T(G_\infty)(K,K) = x(K,K;G_\infty) \neq G_\infty(K)$. Thus, *the lack of convergence implies directly a bias in the computed long term capital stock* as the resourses available to the household are permanently distorted by the numerical procedure.

Note that the convergence criteria in any numerical procedure is *relative.* That is, the algorithm will be "declared convergent" if for $n \geqslant N(\epsilon)$:

$$SUP_K \left| \frac{x(K,K;G_n)-G_n(K)}{G_n(K)} \right| < \epsilon$$

Where ϵ is the tolerance level. Thus, it is possible that $x(K, K; G_n) - G_n(K)$ may be far away from zero even though the numerical procedure has "converged".

So far, we have discussed the implications of the lack of convergence on equilibrium decisions. What can we say about *simulations*? The first step is to define a proper steady state, as simulted paths must converge to a meningful object (i.e. an unconditional moment of an stationary distribution). Since [11] we know that *compactness and continuity* are sufficient to insure the existence of a well behaved steady state (see theorem A.1 in the appendix). Assume that K belongs to a compact set. Since [22], see chapter 5.1, we know that there are curvature conditions associated with F which insure the desired compactness, so the assumptions seems mild. However, in non-optimal economies, the continuity of the equilibrium equations remains an open question. For instance, [7] showed that if $(1 - \tau(K))F'(K)$ is *decreasing* (in K), there is a continuos recursive equilibria. However in our case, as the net rental income is not monotonic, we can't use this result.

Let $g_\tau(k, K) \equiv \pi_t(K_t) + (1 - \tau(K_t))r(K_t)k_t + T_t(K_t)$. If $u(g_\tau(k, K) - x)$ is strictly concave (in k, x) and the feasibility correspondence for the recursive problem is convex, we know from [22], see section A.3 in the appendix, that $V(k, K)$ is stricly concave (in k). Unfortunately, in the present framework, we can not insure the desired properties and thus the value function may not be concave (see section A.3 for a discussion for the stochastic case). Thus, we need to use more general results. From [18] and [3] we know that V has a well defined directional (left) derivative (see section A.2 in the appendix). As, V is not concave, the standard envelope theorem does not hold even if the return function is differentiable (see section A.3 in the appendix). To see why, note that the differentiability of V would have implied that:

$$V'(k, K) = u'(g_\tau(k, K) - x(k, K))(1 - \tau(K))F'(K)$$

At $k = K$, the strict concavity of V implies that $u'(g_\tau(K) - x(K))(1 - \tau(K))F'(K)$ must be decreasing in K, a fact that requires the monotonicity of $(1-\tau(K))F'(K)$, which does not hold by assumption. Thus, any optimal solution must satisfy:

$$u(g_\tau(k, K) - x(k, K)) = \beta V_1^-(k, K)$$

Where V_1^- is the (left) directional derivative with respect to k. As the left hand side is not continuous (as V is not differentiable), the discontinuity is transfered to the left hand side and thus to $x(K, K)$. Since a well defined steady state requires continuity, the simulted paths may not be convergent.

In order to test the (numerical) implications of the *lack of a convergent operator and / or the discontinuity of the equilibrium laws of motion* this paper shows the existence of a *continuous and closed form* recursive equilibrium in an *enlarged state space.* We call this equilibrium notion *Generalized Markov Equilibrium* (GME). The qualitative properties of this type of equilibrium allow us to test the size of the bias as any MSSRE must satisfy the requirements of our definition. Ir order to insure stationarity and compactness, we build a modified version of canonical result due to [9] (see section A.5 in the appendix).

Let K, K_+, K_{++} be the capital stock today, tomorrow and the day after tomorrow, respectively. Then, the first order condition associated with the sequential equilibrium for this economy, for interior solutions, is:

$$u(g_\tau(K) - K_+) = \beta u'(g_\tau(K_+) - K_{++})(1 - \tau(K_+))F'(K_+)$$

One of the main contributions of this paper is to find a function, $H(K, K_+) = K_{++}$ *continuos, unique and with closed form* which satisfy the above equation in an equilibrum path (i.e. when the transfers are budget feasible and the goods market clear). In order to test the implications of our findings on the MSSRE, we can use a result in [3]. The authors showed that even if the net rental income is not monotonic, any solution to the dynamic programming program associated with a MSSRE must satisfy:

$$u(g_\tau(K) - x(K)) = \beta u'(g_\tau(x(K)) - x(x(K)))(1 - \tau(x(K)))F'(x(K))$$

Where the right hand side of the above equation maybe discontinuous. Thus, any MSSRE is a GME as it is restricting K_+ to satify $K_+ = x(K)$.

Suppose that we heuristically find a convergent sequence of functions $\{G_n\}_n$ which is also a MSSRE. In the numerical section below, we provide an example of this type of functions. That is, we avoid the problems associated with $x(K, K; G_n) - G_n(K)$. However, we found that the computed MSSRE converges to a steady state quite far away from the "true" equilibria. The pictures below illustrate the situation at hand: as K_+ is not pin down by any *stationary function* (i.e. x in the MSSRE), the demarcation lines in the plane (K, K_{++}) are pushed towards the boundary of the system during the whole transition. Of course, this is not the case for the MSSRE.

Figure 1 is borrowed from the numerical section of this papers. It depictes the demarcation lines for K, K_{++} given K_+, which are downward sloping and increasing in K_+. Also, the "upper contour" line reflecs the maximal level of K_{++} for a given K, where the boundary reflects the zero consumption pairs. Note that for an arbitrary large n, K_n orbits near the the intersection of the 45 ray with upper contour line as the demarcation curves becomes "sufficiently flat" to revert the monotonic dynamic of the capital stock.

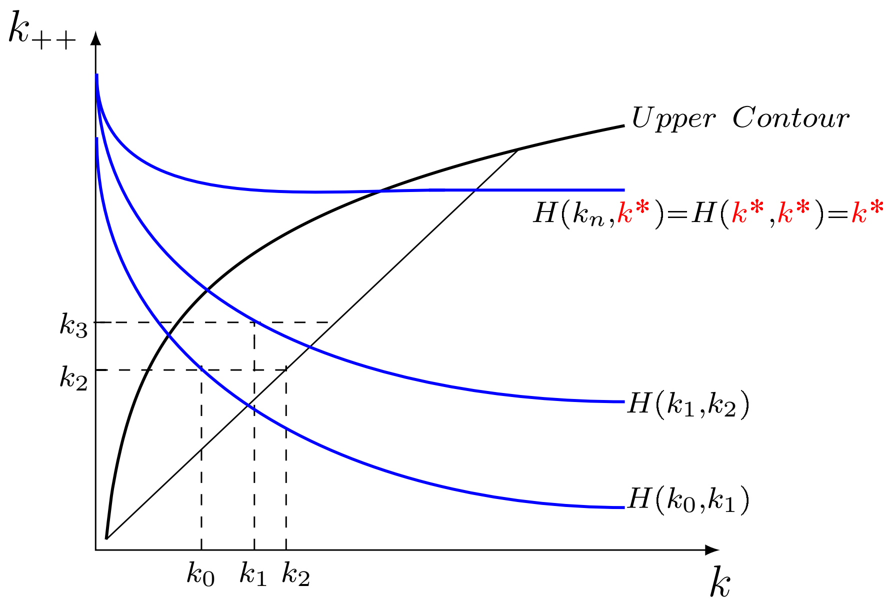

Figure 1: Dynamic Behavior in a Generalized Markov Equilibria (GME)

Figure 2 ilustrates a (numerically) convergent G_*. We know from previous paragraphs that this function will not be continuos, maybe near its intersection with the 45 line. Moreover, any convergent and continuous MSS with $K_n = K_{MSSRE}, n \geqslant N_\epsilon$, will also satisfy $K_n = H(K_n, K_n)$. This last fact is not depicted in Figure 2 for expositional purposes. In particular, we found 2 selections for the closed form GME. However, the state space for one of them is not stationary. Thus, as we only have 1 time independent GME which does not display the same long run behavior of the MSSRE, figure 2 does not represent an equilibrium for the model presented in this paper.

Figure 2 shows the pairs $(k, x(k))$ (in blue) and $(k, x(x(k)))$ (in green) which satisfy the equivalence between the 2 equilibrium types. As we are computing the MSSRE in a finite grid, denoted $\{K_j\}$, we choose to plot points, which are interpolated for expositional purposes. Note that eventually, we can find a pair elements in the grid which satisfy:

$$K_{n+3} = K_{n+2} = Argmax\,\{W(K_{n+1}, K_{n+1})(K_j)\}_{K_j} = Argmax\,\{W(K_{n+2}, K_{n+2})(K_j)\}_{K_j}$$

Where W is the objective function of the Bellman equation in the MSS problem. The expression above is the numerical equivalent to $x(x(K_*)) = K_*$. Note that, as we are dealing with a finite set of points, the continuity requirement is trivial as we can endow the function with the discrete topology. Thus, convergence is achieved numerically *even if the function is not continuous.* Figure 3 depicts a discontinuous mechanism which will be declared convergent by any iterative procedure based on a finite set of (interpolated)

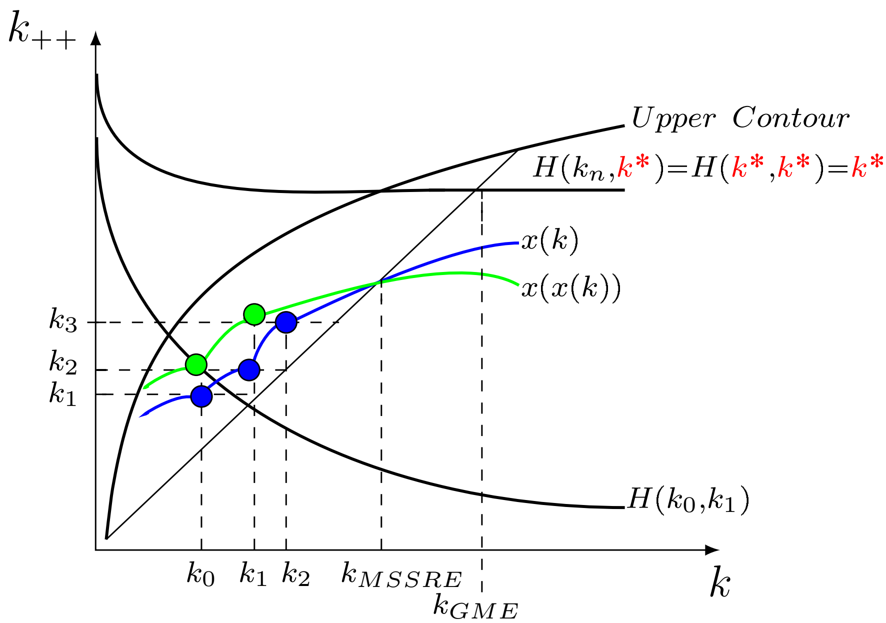

Figure 2: Minimal State Space Markov Equilibria (MSSRE) and GME

points. This figures ilustrates one of the main findings of the paper: a numerically convergent MSSRE which does not have a steady state and a significant bias with respect to the ergodic GME.

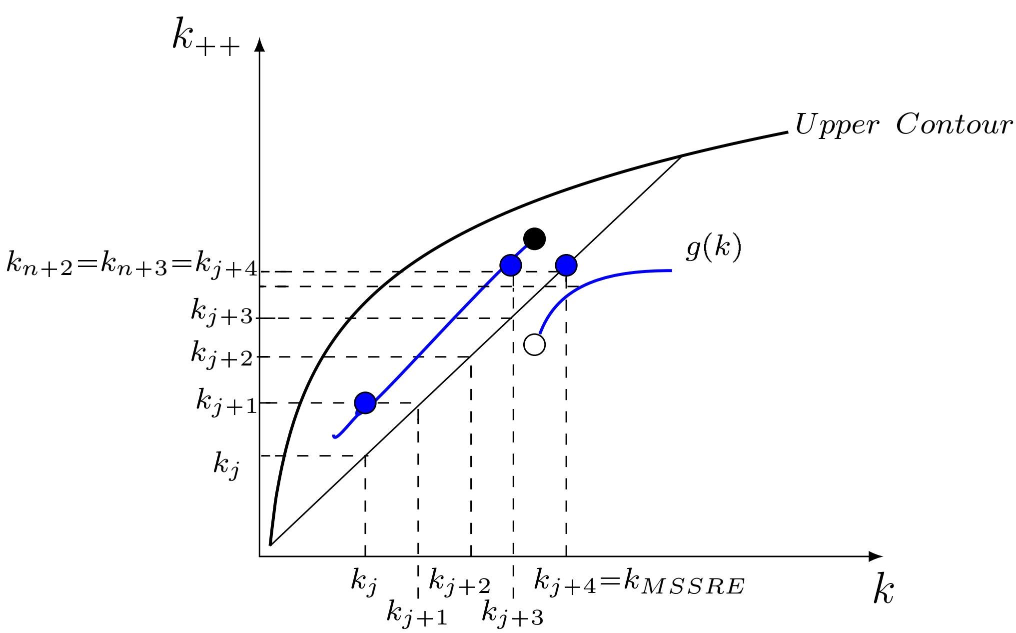

Figure 3: Dynamic Behavior in a Discontinuous MSSRE over an evenly spaced grid $\{K_j\}$

The blue dots are the pairs $(K, x(K, K))$. Note that we are plotting an evenly spaced grid $(k_j, ..., k_{j+4})$ and a possible discontinuity point of the MSSRE (in black dots). The actual image of k_{j+4} *does not belong to the grid*. Moreover, *Argmax* $W(k_{j+4}, k_{j+4})$ is closer to k_{j+4} than any other point in the grid. Thus, as W is typically "bell shaped", the algorithm will pick k_{j+4} as a solution to the maximal problem when the aggregate state is k_{j+4}. Thus, we have $x(k_{j+4}, k_{j+4}) = k_{j+4}$ even though this policy function does not have a steady state. Note that as all MSSRE are a subset of all possible GME, the latter must be discontinuous if we have a bias between the 2 steady states in a convergent solution. If this would have not been the case, the GME would also have a steady state in the same stationary point as the MSSRE. We provide numerical evidence in favor of this hypothesis as we have a GME with 1 ergodic steady state which is not equal to the numerical long run equilibrium in the MSSRE.

To sum up, on one hand, it is possible that numerical simulations converge to $k_{n+3} = k_{n+2}$ simply because the grid is not sufficiently thin. In this case, the steady state of the model is not well defined. On the other, the discontinuity points, if we could identify them, may be far away from the 45 degree line implying that the long run behavior of the MSSRE is accurate. However, due to the exactness, continuity and uniqueness of the GME, we can only be sure about the simulations obtained from this last equilibrium notion. As a GME is a weaker equilibrium notion when compared with the MSSRE, we are confident that *a bias in long run simulations in the presence of a convergent MSS algorithm implies a discontinuity around the 45º line.* This is the most important contribution to the literature of this paper: contrarily to previous results, see for instance [2], we can measure *accurately* the implications of *not* having a convergent method and / or *not* having a well defined (stochastic) steady state. The literature focuses on the sensitivity of the numerical results to different methods, but we were unaware of the size and reasons behind the bias in MSS methods. The results in this paper has direct take away point: as the accuracy of simulations depends on the continuity of the solution, *even if the numerical procedure has been declared convergent using a demanding criteria, simulations may be far away from the exact steady state.* In this sense, the results in [7] seems the natural way to avoid the problems found in this paper as they insure convergence and continuity using a standard convergence criteria based on the SUP norm.

3 The model

3.1 A Stochastic sequential economy with endogenous tax rates

The model is a stochastic version of [19] (section 3.2). Consider a representative agent economy with discrete time, $t = 0, 1, 2....$ Exogenous shocks are markovian and will be denoted z. For the sake of simplicity let us assume that the state space for these shocks is $\{0, 1\}$. An element of the transition matrix will be denoted $p(.,.)$. Let $\{z_t\}$ be a sequence of shocks and Z^t the set of histories up to time t, being a typical element z^t. Using standard results (see [22], Ch. 8) it is possible to define, for any $z_0 \in \{0, 1\}$, a stochastic process $(\Omega, \sigma_\Omega, \mu_{z_0})$ on Z^∞.

As in this section we are dealing with a sequential economy, k denotes the supply of capital (services) and K its demand. There is a unique decreasing return to scale firm which only uses capital as input and its technology is characterized by $y_t = A(z_t)f(K_t)$ with $f' > 0$, $f'' < 0$ and $f(0) = 0$ as usual. The firm is owned by the consumer as she is endowed with $k_0 > 0$ units of capital. Thus, the agent has two sources of current income derived from her endowment: benefits, denoted by π_t, and rents from capital, denoted by $r_t k_t$. Besides, the flow of taxes paid and transfers received is $\tau(K_t)r_t k_t$ and T_t respectively. Note that the tax rate depends on the stock of capital. In particular, it is given by a piecewise linear continuous and decreasing function (see [19], page 87 for details).

The problem faced by the consumer is to choose a pair of functions $c : Z^\infty \longrightarrow \mathbb{R}_+$ and $x : Z^\infty \longrightarrow \mathbb{R}_+$ that solves the following problem:

$$\max_{\{c,x\}} \sum_t \sum_{z^t \in Z^t} \gamma^t u(c(z^t))\mu_{z_0}(z^t) \tag{2}$$

s.t.

$$k(z^t) = x(z^t) + (1-\delta)k(z^{t-1}) \tag{3}$$

$$c(z^t) + x(z^t) \leqslant \pi(z^{t-1}) - (1-\tau(z^{t-1}))r(z^t)k(z^{t-1}) + T(z^t) \tag{4}$$

$c(z^t) \geqslant 0, x(z^t) \geqslant 0$ for any $z^t \in Z^t$, z_0 and $k_0 > 0$ given, $\delta \in [0,1]$ is the depreciation rate and $\gamma \in (0,1)$ the discount factor.

Note that we are restricting the maximal random variables (c, x) to take values on $\mathbb{R}_+$. This restriction will be relevant for the recursive representation of the sequential equilibria as boundary conditions will be critical to prove existence of a stationary state space. In what follows $\tau(z^{t-1})$ stands for $\tau(k(z^{t-1}))$ or abusing notation $\tau(k_t(z^{t-1}))$.

That is, the tax rate affects the rents obtained from capital holdings at time t, which is in turn affected by the information contained in z^{t-1} because $k_t(z^{t-1}) = x_{t-1}(z^{t-1}) + (1-\delta)k_{t-1}(z^{t-2})$. A similar argument can be used to understand $r(z^t)$ because the agent knows the clearing condition for the market of factors and the optimality condition for the firm to be described below.

The problem of the firm is standard. Taking r_t as given it solves:

$$\max_{K_t} A(z_t)f(K_t) - r_t K_t, \qquad \text{for any } z_t \in \{0,1\}. \tag{5}$$

Observe that the optimality of the firm implies $r_t = A(z_t)f'(K_t)$. The Government simply transfers to the consumer the tax revenues:

$$T = \tau(z^{t-1})r(z^t)k(z^{t-1}). \tag{6}$$

Finally, goods and factor markets clear:

$$c(z^t) + x(z^t) = A(z_t)f(K_t) \qquad \text{Goods Market}$$
$$k(z^t) = K_{t+1} \qquad \text{Capital Market}$$

where both equations hold for any $z^t \in Z^t$.

Note that in equilibrium, the optimality condition of the firm and the market clearing equation for capital holdings implies $r_t = A(z_t)f'(k(z^{t-1}))$ which in turn implies $r_t = r(z^t)$ as claimed. Further, both market clearing conditions imply $c(z^t) + x(z^t) = A(z_t)f(k(z^{t-1})) = y(z^t)$ as expected.

We can now define the sequential equilibrium for this economy:

Definition 1 A Sequential Competitive Equilibrium for this economy is composed by a triad of functions z^t measurable functions (x, c, r) such that:

- Given r, (x, c) solve the Maximization problem of the household.
- For each z^t, given $r(z^t)$, $K(z^t)$ solves the problem of the firm.
- For each z^t, Goods and Capital markets clear.
- For each z^t, the Government runs a balanced budget, equation (6).

3.2 Equilibrium Equation

In this case, the solution to the model can be characterized by the equilibrium Euler equation, which can be obtained by putting the optimality condition for the firm, the budget constraint for the Government and the market clearing conditions into the optimality condition for the consumer.

Assume that $u(c) = \ln(c)$ and $\delta = 1$. Then, the equilibrium equation is given by:

$$\tfrac{1}{C_t} = \gamma \sum_{z_{t+1}=0,1} \frac{A(z_{t+1})p(z_t, z_{t+1})(1 - \tau(K_{t+1}))f'(K_{t+1})}{C_{t+1}}, \tag{7}$$

With constrains given by

$$K_{t+1} = A(z_t)f(K_t) - C_t. \tag{8}$$

Note that the market clearing condition for capital implies that *given* z^t the demand for capital K_{t+1} does not depend on the realizations of the exogenous shock at $t + 1$.

Hence, by replacing C_{t+1} in (7) with its expression obtained from (8) and after some algebra we can rewrite (7) in the following way:

$$\overbrace{\underbrace{\frac{1}{\gamma(A(z_t)f(K_t)-K_{t+1})(1-\tau(K_{t+1}))A(z_t)f'(K_{t+1})}}_{c}}^{c} = \frac{\overbrace{A(0)p(z_t,0)}^{c_1}}{\underbrace{A(0)f(K_{t+1})}_{d_1}-K_{t+2}} + \frac{\overbrace{A(1)p(z_t,1)}^{c_2}}{\underbrace{A(1)f(K_{t+1})}_{d_1}-K_{t+2}} \tag{9}$$

One of the purposes of this paper is to find an equation $\Psi : X \longrightarrow X$, where X is an appropriately defined state space and Ψ is a function that maps $x_t \longmapsto x_{t+1}$ with (x_t, x_{t+1}) satisfying equation (9) for any t.

Notice that by standard arguments, by fixing $\delta = 1$ and $f(0) = 0$, K_t stays in $[0, K^{UB}]$ (see [22], Ch. 5) for any t.

Let $X = [0, K^{UB}] \times [0, K^{UB}] \times \{0, 1\}$. With this state space Ψ becomes a vector valued function of the form $x_t \longmapsto (\Psi_1(x_t), \Psi_2(x_t), \Psi_3(x_t))$ with $x_t = (K_t, U_t, z_t)$.

Let $\{z_n\}$ be a realization of $(\Omega, \sigma_\Omega, \mu_{z_0})$. Then, it is possible to define each coordinate in the image of Ψ as follows:

$$\begin{aligned} K_{t+1} &= \Psi_1(x_t) = U_t \\ z_{t+1} &= \Psi_3(x_t) = \{z_n\}(t+1). \end{aligned}$$

In order define Ψ_2 we could use (9). Notice that (9) takes the form

$$c = \frac{c_1}{d_1 - U_{t+1}} + \frac{c_2}{d_2 - U_{t+1}}, \tag{10}$$

or equivalently,

$$c(d_1 - U_{t+1})(d_2 - U_{t+1}) = c_1(d_2 - U_{t+1}) + c_2(d_1 - U_{t+1}). \tag{11}$$

Due to the fact that this is just a quadratic equation we can get U_{t+1} as a *continuous function* of the parameters, namely:

$$U_{t+1} = \frac{\pm\sqrt{(-d_1c - d_2c + c_1 + c_2)^2 - 4c(d_1d_2c - c_1d_2 - c_2d_1)} + (d_1 + d_2)c - c_1 - c_2}{2c}. \tag{12}$$

Equivalently:

$$U_{t+1} \equiv g(d_1, c, d_2, c_1, c_2)$$

It is important to observe that (12) gives at most *2 different mechanisms* [2] , each of them characterized by a different root of (12). Furthermore, note that $c(K_t, U_t, z_t)$, $d_1(U_t)$, $d_2(U_t)$ and the rest of the parameters in (12) depend on z_t . Thus, Ψ_2 is given by:

$$U_{t+1} = g(d_1, c, d_2, c_1, c_2) \equiv \Psi_2(x_t).$$

Note that once we start iterating the sytem, it is possible that simulations go outside X. The continuity of Ψ_2 (on K_t and U_t), provided that the state space is well defined across any possible path, seems automatic. It suffices to verify the continuity of C, d_1, d_2 (on K_t and U_t), which is trivially satisfied. However, if we iterate forward equation (12), the restrictions on d_1, c, d_2, c_1, c_2 in order to keep U_{t+1} in $\mathbb{R}$ may affect the empirical performance of the model as the set of parameters (i.e. β, $p(.,.)$, etc) can't be freely choose in the calibration / numerical estimation procedure. Moreover, even if we could find an empirically meaningful parameter set, any solution to (12) may imply a negative consumption level or capital stock. Of course, due to the *log* preferences, these solutions will not be optimal. Thus, we have to find a procedure in order to rule out solutions outside $\mathbb{R}_+$ and that imply a non-positive consumption level. In the numerical section, we adapt a canonical result due to [9] in order to verify that the state space is well defined along equilibrium trajectories. Given the quadratic structure in (12), the stationarity (i.e. time independence) of the state space is sufficient to insure both compactness and contunuity of the recursive mechanism.

3.3 Generalized Markov Equilibrium

The previous section describes a recursive mechanism based on an *enlarged* state space X. In particular, we wrote K_{t+2} in terms of (K_t, K_{t+1}, z_t):

$$K_{t+2} = g(K_t, K_{t+1}, z_t).$$

The mechanism, g, is *explicit* and, even more, continuous (of course, this representation has economic content if we can assure that the discriminant in g is positive under reasonable parameterizations for any $x \in X$ and the boundary conditions on endogeneous variables are satisfied).

[2] Note that (9) implies that this model does not have a trivial solution at $K_t = 0$ as $u = \ln$ and investment is not allowed to be negative. This fact in turn implies that the parameters in (9) are all bounded away from 0. Of course, in order to have two non-trivial solutions it suffice to impose conditions on the discriminant of (12)

We can now define a Generalized Markov Equilibria.

Definition 2: Generalized Markov Equilibrium (GME)

A GME is a *correspondence* $\Psi : X \rightarrow X$ with X compact such that for any $x \in X$, the vector $(x, \Psi(x))$:
i) satisfies the optimality conditions for the household problem, equation (2) s.t. (3) - (3).
ii) The firm solves (5)
iii) Markets clear
iv) The public sector runs a balanced budget. That is, equation (6) holds.

In section 3.2 we show that, if we can insure the existence of a well behaved state space, the sequential version of the model presented in this paper has a GME representation. Moreover, Ψ may even have 2 continuous selections. Let Ψ_i be any of the 2 possible selections. Using standard results (see [22]), we can show that $P_{\Psi_i}(x, A)$ defines a Markov kernel with $P_{\Psi_i}(x, .)$ being a probability measure for any $x \in X$ and $P_{\Psi_i}(., A)$ being a measurable function for any $A \in Borel(X)$. An invariant measure is any fixed point of Ψ_i. Call one of the possible many fixed point μ_i.

Let Ψ_i^j be any numerical approximation to Ψ_i and $P_{\Psi_i^j}(x, A)$, μ_i^j the associated Markov kernel and invariant measure respectively. Since [20], it is known that even if Ψ_i^j converge to Ψ_i, the simulations obtained from Ψ_i^j may differ from the exact ones, generated using Ψ_i. If Ψ_i is equicontinuous and defined over a compact state space, these authors showed that numerical simulations will match the exact long run behavior of the model. However, equicontinuity is associated with very restrictive properties for non-optimal economies as noted in [7]. If Ψ_i is not continuous / equicontinuos, [15] provided sufficient conditions which insure that numerical simulations replicate the actual model. Unfortunately, these conditions depend on the cardinality of Z, the set containing exogenous shocks, and will not hold in this framework.

The virtue of this paper is that it allows us to circunvent the mentioned problems. On one hand, we show that *a GME exist* for the problem at hand and thus, it is possible for us to compute it. Moreover, using (7) and (8), we show that Ψ_i has a *continuous closed form representation*, which in turn eliminates the problem associated with the lack of convergence of numerical simulations, provided that we can find a suitable state space.

The (numerical) cost of this representation is the enlargement of the state space with respect to the natural one (i.e. (K_t, z_t)). As we have a closed form solution, these costs are more than compensated by the accuracy of simulations. As discussed in [12], enlarging the state space might provide a recursive representation. Unfortunately, the results in that paper does not address the continuity of the mechanism; an aspect that has severe consequences for the steady state of the model as discussed in [9].

This paper shows that it is possible to obtain a continuous selection from a correspondence-based recursive representation. After taking care of the boundary conditions, we can insure the compactness of the state space. Coupled with the continuity of the mechanism, Ψ_i, we can show existence of μ_i using canonical results in [11]. See section A.1 in the appendix for a detailed discussion about the exustence of invariant measures in compact spaces.

We can use these results to simulate the model. As $U_t := K_{t+1}$, we have now the following iterative system:

Take first an arbitrary initial condition (K_0, U_0, z_0) and a drawn $\{z_n\}$, then

$$\begin{aligned} K_{t+1} &= U_t \\ U_{t+1} &= g(K_t, U_t, z_t), \end{aligned}$$

provides a sequence $\{X_n\}$. Such a sequence defines a Feller mechanism, with compact state space X.

From [20] and [21], we know that $P_{\Psi_i}(x, .)$ has an ergodic invariant measure if Ψ_i is equicontinuos. The quadratic structure in (12) insures that the compactness of the state space and the interiority of solutions are sufficient (if they are satisfied jointly, of course) to guarantee ergodicity. In section 4.2 we will define an operator which allows us to find a state space that it is compact and that insures that capital and consumption remains positive along equilibrium paths. These facts in turn, guarantee that the derivatives of Ψ_i are finite, which in turn implies equicontinuity. Provided that μ_i is ergodic, the process $\{K_t\}$ has a well defined invariant measure as well. Moreover, using standard results on laws of large numbers for markov processes (see [24]), it can be shown that choosing an appropriate initial condition suffices to guarantee that:

$$\frac{\sum_{t \in 0,\ldots,T} h(X_t)}{T} \quad \text{converges almost surely to} \quad E_\mu(h),$$

Where h is a X-measurable function and μ is one of the possibly many ergodic invariant measures described above.

Finally, note that U_{t+1} is measurable with respect to z^t, which in turn implies that K_{t+2} is measuble with respect to the same filtration. As $Z^t \subset Z^{t+1}$, the measurability requirements in definition 1 are satisfied. This is the cost of working with a Markov structure: we are loosing memory inherited from the sequential equilibrium, a fact which may affect the empirical performance of the model as noted by [16].

3.4 Minimal State Space Recursive Equilibrium

This paper deals with global methods, which are widely used in practice. The literature has also made substantial progress in the desing of local methods. There is a clear trade-off between these 2 options: while the former is able to replicate a more flexible dynamic behavior, the latter is capable of dealing with large scale models. Any researcher choosing a global method has to deal with the limitations implied by the numerical burden associated with the solution of a considerable number of non-linear equations. Thus, it is natural to choose the mininal possible number of states as this option reduces significantly the main disadvantage of global methods.

In this sense, it is critical to understand the limitations of Minimal State Space Recursive Equilibrium (MSSRE) methods. The MSS version of the model described above can be written as follows:

$$V_n(k, K, Z; H_j) = Max_{y \in \Gamma(k,K,Z)}\, u(g_\tau(k, K, Z) - y) + \beta \sum_{Z'} V_{n-1}(y, H_j(K, Z), Z'; H_j) p(Z, Z') \tag{13}$$

Where the feasibility correspondence is given by:

$$\Gamma(k, K, Z) = [y \in \overline{K}; 0 \leqslant y \leqslant \pi(K, Z) + (1 - \tau(K, Z)) r(K, Z) k + T(K, Z)]$$

Capital is allowed to fluctuate in a compact set, $[0, K^{UB}] = \overline{K}$. The function g_τ represent disposible income and is defined by:

$$g_\tau(k, K, Z) \equiv \pi(K, Z) + (1 - \tau(K, Z)) r(K, Z) k + T(K, Z)$$

Where $\pi(K, Z)$ and $\tau(K, Z)$ are defined in (3) and $T(K, Z)$ in (6). The policy function for (13) is given by $h_{n-1,j}(k, K, Z)$, which belongs to the set defined below:

$$argmax \left\{ u(g_\tau(k, K, Z) - y) + \beta \sum_{Z'} V_{n-1}(y, H_j(K, Z), Z', H_j) p(Z, Z') \; s.t. \; y \in \Gamma(k, K, Z) \right\}$$

Note, remarkably that: i) the household take a guess at the evolution of the aggregate states using a *perceived law of motion* denoted H_j. ii) The value and the policy function in the dynamic programming problem have to converge in j, which is associated with the rational expectation nature of the problem (i.e. the perceived and the actual law of motion must be equal when $k = K$), and in n, that is guarenteed by the contractive nature of the Bellman operator in (13). iii) The dependence of disposible, $g_\tau(k, ., .)$, on prices, $r(., .)$, justifies the presence of *equilibrium states* which are represented by capital letters. In particular, they affect the household problem through the firm's decisions, given by

(5), and market clearing conditions which are contained in the definition of recursive competitive equilibrium, which is given below.

Definition 3 Minimal State Space Recursive Equilibrium (MSSRE)

A MSSRE is a *value function* V_*, a *policy function* $h_{*,*}$ and a *perceived law of motion* H_* such that:
i) the household solves equation (13) obtaining $V_*(k, K, Z; H_*)$ and $h_{*,*}(k, K, Z; H_*)$ for any feasible state k, K, Z.
ii) The firm solves (5)
iii) Markets clear. That is, $k = K$
iv) Expectations are fulfilled. That is, $h_{*,*}(K, K, Z; H_*) = H_*(K, K, Z)$ for any (K, Z)
v) The public sector runs a balanced budget. That is, equation (6) holds.

In order to undertand the connection between the existence of a MSSRE and its computation, we must characterize it. Even under strong curvature and smoothness assumptions on the return function u, which are all satisfied imposing the parametrizations used in sections 3.1 and 3.2, even if we assume the continuity of the feasibility correspondence Γ, for an interior optimal solutions, $h_{*,j}\left(., K, .; H_j\right) \in \Gamma\left(., K, .\right)$, we can't use the envelope theorem in [22]. The arguments used in section 2 hold *mutatis mutandis.* In particular, Benvenisite and Scheinkman envelope theorem (see [22] page 266, Th. 9.10) coupled with the strict concavity of V_n (in k) (see [22] page 265, Th. 9.8) would imply that V_n' is decreasing in k when $k = K$, which will not hold globally as $f'(K)(1 - \tau(K))$ is not monotonic. Critically, the feasibility correspondence $\Gamma(k, K, Z)$ is not convex (see section A.3 for a detailed discussion).

Fortunatelly, using lemmas 3.3. and 3.4 in [3] we know that any solution to the dynamic program must satisfy the "classical" Euler equation and, thus, it can be characterized (see section A.4 for a detailed discussion). Formally, a solution to the dynamic programming problem in definition 3 for any pair of individual states (k, Z) and given the aggregate level of capital K must satisfy:

$$u'\left[g_\tau(k, K, Z) - h_{*,j}\right] = \gamma E_Z \left\{u'\left[g_\tau(H_j, h_{*,j}) - h_{*,j}(h_{*,j})\right] A f'(H_j)(1 - \tau(H_j))\right\} \qquad (14)$$

Where the dependence of $h_{*,j}$ on (k, Z) for each K and of H_j on (K, Z) have been omitted for expositional purposes. Also, equation (14) does not include the equilibrium version of g_τ, the disposible income, as condition iv) in the definition 3 may not hold in this model, even when $k = K$.

Note that (14) defines a mapping T from H_j to $h_{*,j}$. In fact, it is easy to see that any fixed point on this map is a MSSRE. Define the function space B on $K \times Z \equiv S$ as follows:

$$B(S) = \{H(s) \text{ such that } H : S \to K \text{ with } 0 \leqslant H(s) \leqslant A(Z)f(K),\ H \text{ measurable}\}$$

That is, a MSSRE is a fixed point in the functional T as the measurable maximum theorem insures that $h_{*,j} \in B$ when $k = K$. Any attempt to prove the existence of a fixed point in a function space has to circunvent the problem associated with the lack of sufficient conditions which insure a convex graph in tractable frameworks. That is, $T(H_j)$ may not be convex for models with a finite number of agents or finite shocks (see [15] for a detailed discussion). Thus, the literature has turned to the lattice dynamic programming framework because it works in non-convex models. See section A.4 for a review of the results in this literature relevant for the model presented in section 3.1.

Moreover, contrarily to the Fan - Gliksberg theorem, lattice dynamic programming gives us a constructive fixed point theorem which naturaly generates an algorithm. In fact, the numerical procedure in [17] can be proved to be convergent endowing B with an order topololgy if T *is a monotone operator*; which in turn insures the existence of a MSSRE. That is, in order to prove the existence of a MSSRE *and* the convergence of the algorithm in [17] for any $H_j^{'} \geqslant_* H_j$ we must have $T(H_j^{'}) = h_{*,j}^{'} \geqslant_* h_{*,j} = T(H_j)$ where $\geqslant_*$ is the pointwise order in B.

In order to prove the desired properties in T we can borrow from [1] and [7]. We present the relevant theorems in section A.4. The former proved that it is required to show that $V_*(k, K, Z; H_j)$ has increasing differences (see section A.4 in the appendix) in $(k; K)$ for each (Z, H_j) (lemma 12 and theorems 3 to 6). This condition, in turn, is equivalent to show that $V_{*,1}^{-}(k, K, Z; H_j) = u'(g_\tau(K) - h_{*,j}(K))(1 - \tau(K))r(K)$ is *increasing* in K, where the dependence of $V_{*,1}^{-}$ on $(k, Z; H_j)$ has been omitted in the right hand side of the equation and $V_{*,1}^{-}$ is the left derivative of V_* with respect to k which is finite (see sections A.2 and A.4 in the apprendix). Note that $(1 - \tau(K))r(K)$ is decreasing in K if τ is increasing and undefined otherwise.Thus, as τ is decreasing by assumption, the results in [1] does not hold.

[7] showed that if $u'(g_\tau(K) - h_{*,j}(K))$ is decreasing in K when $k = K$, it is sufficient to assume that τ is increasing to induce an order structure using an operator based on (14). As τ is decreasing by assumption, we have shown that even if $u'(g_\tau(K) - h_{*,j}(K))$ is monotonic in K, as $(1 - \tau(K))r(K)$ is undefined, we cannot have an order structure for this model.

This last fact implies in turn that it is not possible to insure that a sequence of function $\{H_j\}_j$ converging to H_* will "hit" $h_{*.*}$ as required by definition 3. Moreover, *any numerical procedure based on iterations through* T using the uniform metric, as the one described in [17], *cannot be proved to be convergent to a MSSRE* as the induced topology is stronger than the order topology. Thus, $SUP\,|H_{*,j} - h_{*,j}|$ maybe arbitrarily large, a fact which can cause a severe bias in the numerical simulations as discussed in section 2.

4 A numerical exploration

The results in section 3 provide a unique oportunity to test the predictive power of MSS methods. As any MSSRE must satisfy equations (7) and (8), the simulations generated by it must converge to one of the possible multiple ergodic distributions obtained using a GME.

In order to perform this test, we present a standard recursive competitive MSS algorithm with 2 different "updating" rules. The first does not numerically converge to a fixed point between the perceived and actual law of motion and the second does, implying that in the latter case we are dealing only with the effects of a discontinuous equilibrium. Then, the policy functions are simulated and the results compared with those obtained from equation (12). In order to insure that the exact GME has a state space which generates a pair of equilibrium random variables (c, x) taking values in the in the non-negative real numbers, we adapt a theorem from [9]. We found that only 1 mechanism has a well defined state space. Thus we know that any MSSRE is a GME, which is also unique. So, any simulation obtained from the former, must match the latter if this equilibrium exist and it is continuous.

As neither continuity nor existence hold in the MSSRE for the model presented in section 3, we found a significant deviation with respect to the true equilibrium which, in turn, affects the long run distribution of capital. These findings provides evidence in favor of the results in [2] and [10] which suggest the importance of theoretical results in the recursive numerical literature. That is, without sufficient conditions that insure the equivalence between numerical and actual simulations of the model, *a convergent algorithm does not guarantee by itself the absence of biases.*

From a qualitative perspective, the ergodic simulations imply that the true long run capital stock is fluctuating near a zero tax rate. Thus, according to recent findings (see [23]), the decentralized equilibrium is not optimal and there is a scope for interventions. *Using the simulations obtained from the MSSRE, contrarily, the economy may be near the optimal tax rate and we would conclude the opposite.*

4.1 MSS Algorithm

We compute a MSSRE as described in definition 3 using the operator T, which in turn follows from equation (14). It is standard in the literature (see for instance, [17]) to pick an arbitrary function H_0 from B and look for uniform convergence. However, as mentioned in section 3, theoretical results do not support such a strong convergence notion. If $(1-\tau)F'$ is increasing in K, it is only possible to show that any iteration starting from a lower or upper bound on T (i.e. $H \in B$ such that $H \leqslant T(H)$ or $T(H) \leqslant H$ respectively) will converge in the order topology. That is, take a sequence of increasing functions generated iteratively from T, $\{H_j\}$ with $H_{j+1} = T(H_j)$. We say that $H_j \rightarrow_{\geqslant *} H_*$, meaning $\{H_j\}$ converge in the order topology to H_*, if for any j, $H_j \leqslant H_*$ and $H_* \in B$. If $(1-\tau)F'$ is decreasing in K, the convergence will be uniform in the standard sup norm. Unfortunately, as τ is decreasing and F strongly concave, we showed in section 3.4 that T is not a monotonic operator and thus it is not possible to generate a convergent sequence of functions using T.

The discussion in the above paragraph is enterily theoretical. We do not have *known sufficient conditions* which insure the convergence to a MSSRE using T. However, we found *numerically* a fixed point for T in the sup norm. In particular, the proceadure described below was found to be convergent using the sup norm for acceptable relative error levels (in the order of 10^{-2})

$$H_0 \rightarrow_{Equation(13)} h_{*,0} \rightarrow_{Definition\ 3} \rightarrow H_1(K,Z) = G^1(H_0, h_{*,0})(K,Z) \rightarrow (...)$$

Where the first $\rightarrow$ means that we are solving equation (13) using H_0 as a guess for the perceived law of motion. The second $\rightarrow$ stands for the fact that we are computing the policy function $h_{*,0}$ along the equilibrium path according to definition 3. The last $\rightarrow$ implies that we are updating the perceived law of motion for aggregates states. The functional G is an updating rule. We use 2 different types of them:

- $G_1^j = \alpha H_j + (1-\alpha) H_{j-1}$, with $\alpha \in (0,1)$
- $G_2^j = \sum_{i=0}^{j} H_i / j$.

The last one was found convergent, that is: $n \geqslant N(\epsilon)$ imply $|G_2^n - h_{*,n}| < \epsilon$. Finally, $\rightarrow$ (...) means that we are starting the loop again if convergence using the sup norm is not achieved.

4.2 Stationary GME

For any arbitrary state space X, with typical element $(K, K_+, Z) \in X$, equation (12) may imply that $K_{++} \notin \mathbb{R}_+$ (i.e. K_{++} may be an imaginary or negative number). Let Z_L be the smallest possible shock. Note that we may have $A(Z_L)f(K) < K_+$ and / or $A(Z_L)f(K_+) < K_{++}$, which in turn imply that consumption is negative.

In order to solve these problems, we modify theorem 1.2 in [9]. The authors showed that, given the compactness of the sequential equilibria, it is always possible to find a stationary sate space $J \subseteq X$ for the markov equilibrium associated with any root of equation (12) using the following iterative procedure:

$$C_1 = \{x_0 \in X \mid \Psi(x_0) \cap C_0 \neq \emptyset\}$$

Where $C_0 \equiv X \subset \mathbb{R}^3_+$. Moreover, for $n \geqslant N$, $C_n \to J$. If *the sequential equilibria is compact*, J is non-empty and compact (see section A.5 in the appendix). As some roots of equation (12) may not be a real number, we can use this operator in order to keep those mechanisms, if any, that are self-contained in $\mathbb{R}^3$. Moreover, the authors showed that for any $x \in J$, $\Psi(x) \cap J \neq \emptyset$, which in turn implies that this mechanism can be iterated forward if J is non-empy. As this set is time invariant, it is a state space for Ψ.

However, it is possible that, for some $x \in J$, the vector (c^L, c^L_+, K_{++}) contains a negative number, where $c^L = A(Z_L)f(K) - K_+$ and $c^L_+ = A(Z_L)f(K_+) - K_{++}$. In order to circumvent these problems, we use the following modified operator:

$$C_1 = \{x_0 \in X, A(Z_L)f(K) \geqslant K_+ \mid \Psi(x_0) \cap C_0 \neq \emptyset, A(Z_L)f(K_+) \geqslant \Psi_2(x_0)\}$$

Where $\Psi_2(x_0) = K_{++}$ is the second coordinate in the image of the vector valued function which defines the GME. The operator above generates a sequence of sets in $\mathbb{R}^3_+$ with non-negative consumption levels which converge to a possible empty set J. We are interested in finding 1 mechanism Ψ from equation (12) which generates a non-empty state space J.

4.3 Numerical Simlations

We now turn to measure the numerical bias. Section 4.1 described the algorithm tipycally used to compute a MSSRE. The numerical procedure associated with the simulation of a GME is contained in the discussion of sections 3.3 and 4.2.

The task is to compute definitions 2 and 3 using a concrete tax function based on the model in [19], a standard algorithm borrowed from [17] and the refinement for the GME described in section 4.2. In particular, τ is decreasing in K. Thus, the discussion in section 3.4 implies that the operator T is not monotonic and, consequently, it is not possible to prove that a numerical procedure based on iterations using T will converge to a MSSRE. The rest of the parameters are contained in the table below. We are carefully following the preferences and technology structure in [19]. However, as this model is non-stochastic, we are setting the values for the exogeneous shocks in set Z and transitions probabilities p_{LH} and p_{HL} in order to insure a well defined steady state for the GME.

$y = A(Z)f(K) = e^Z K^{1/3}$	$Z_H = 0.2$
$u(c) = ln(c)$	$Z_L = 0.1275$
$\delta = 1$	$p_{LH} = 0.5$
$\beta = 0.99$	$p_{HL} = 0.3$

Table 1: Parameters

The table below contains the results of simulating the proceadures described in section 4.1 (MSSRE) and 3.3 (GME). The parameters used are listed in Table 1. We refine the mechanisms for the GME using the operator defined in section 4.2. We found that for the negative root $J = [0.01, 1.50]$ and for the positive root $J = \emptyset$. Thus we will only report $\Psi_{NR} \equiv \Psi$, where "NR" stands for negative root.

Model	Mean	STD	CV
Ψ	1.1976	0.0079	0.0066
MSSAvg, $K_{UB} = 0.6$	0.4058	0.0117	0.0289
MSSCes, $K_{UB} = 0.6$	0.2662	0.0106	0.0400
MSSCes, $K_{UB} = 1.5$	0.3098	0.0134	0.0431

Table 2: Simulation Results. Statistics for aggregate capital

Where STD stands for standard deviation and CV for the coefficient of variation (standard deviation / mean). The "empirical" distributions are constructed as follows: take an arbitrary initial condition. Simulate a path of 5000 observations for aggregate capital. Store the last 1000 observations. Then, the computed distribution is taken from the relative frequency of 25 grid positions out of these observations. The proceadure is repeated for any of the 4 listed distributions.

We report 3 different solutions for the MSSRE, which differs in the updating rule discussed in section 4.1 and in the upper bound (K_{UB}) of the grid. The first one called "MSSAvg", which stands for "average", is not convergent and thus it contains the 2 sources of biasses: the lack of a steady state and the lack of convergence. Unfortunately, we can only compute the last one. When we expand the state space, in order to make it comparable with J, the cesaro updating is not convergent. Below we report the bias associated with the first and the last case.

Model	Min-Max Error	Min-Max Rel. Error	$A(Z_{LB})F'(K^*)$	Bias
MSSAvg, $K_{UB} = 0.6$	$[0.1062, 0.4248]$	$[0.9696, 41.3]$	0.6908	0.1834
MSSCes, $K_{UB} = 0.6$	$[-0.0059, 0.0118]$	$[0.072, 0.1361]$	0.9151	0.0027
MSSCes, $K_{UB} = 1.5$	$[0.0596, 0.3129]$	$[0.3460, 0.8994]$	0.8270	0.1540

Table 3: Lack of Convergence: Implications for the accuracy of simulations

We define an error as the diference between the perceived ($H_{*,j}$) and actual ($h_{*,j}$) law of motion for capital. The columns in the table contains the [minimum - maximum] relative and absolute errors across iterations j. The relative error determines the convergence of the algorithm. Note that only the cesaro updating procedure with a grid of $[0.01, 0.6]$ converged.

The absolute error is used to compute the distortion generated by the algorithm. If we take as a reference value the mean of the capital stock under each procure, denoted K^*, the lack of convergence of the algoritm implies a distortion of $(h_{*,j}(K, K, Z) - H_{*,j}(K, Z))A(Z)F'(K)$ in the equilibrium budget constraint $c + x = F(K)$. That is, on average, the MSSAvg procedure implies that the household receives 0.1834 more units of the consumption good due to the lack of convergence of the algortim. Thus, as the agent is "wealthier", capital stock is higher when compared with the accurate solution among MSS algorithms (i.e. MSSCes, $K_{UB} = 0.6$).

The numerical solutions in Table 2 has a significant bias, as measured by the difference in mean with respect to the ergodic distribution. The table below presents the relative deviations.

Model	Relative Mean	Relative CV
MSSAvg, $K_{UB} = 0.6$	0.34	4.38
MSSCes, $K_{UB} = 0.6$	0.22	6.06
MSSCes, $K_{UB} = 1.5$	0.25	6.53

Table 4: Relative Bias

Where "Relative" stands for $Mean(MSSAvg, K_{UB} = 0.6)/Mean(\Psi)$, etc. From Table 4 it is clear that the mean of the ergodic accurate distribution (as measured by the GME) is way above the mean generated by any numerical approximation of the MSSRE. On the contrary, the dispersion is signifficantly below. Thus, despite the fact that the algorithm for the MSSRE converge for the case of $MSSCes, K_{UB} = 0.6$ using a strong criteria (i.e. the sup norm and a tolerance level of 0.075 for the relative error), the numerical distribution will present a severe bias with respect to the distribution *that we know is well defined*.

The results described above point out to the relevance of a well defined steady state (i.e. a fixed point of $P_H(K, Z; .)$, where P is the markov kernel defined in section 3.3 but constructed using the perceived law of motion for the MSS, H). From section 3.4 and figure 3 we know that the discontinuity of $V_{*,1}^-$ plays a central role in this fact. Below we show the (numerical) derivative of the value function for the $MSSCes, K_{UB} = 1.5$ when $k = K$. We choose this solution as the state space is comparable with J.

Where the blue line represent the derivative for the low shock. Even though the figure depictes the expected convex shape for a concave function, it has sevelaral jumps,

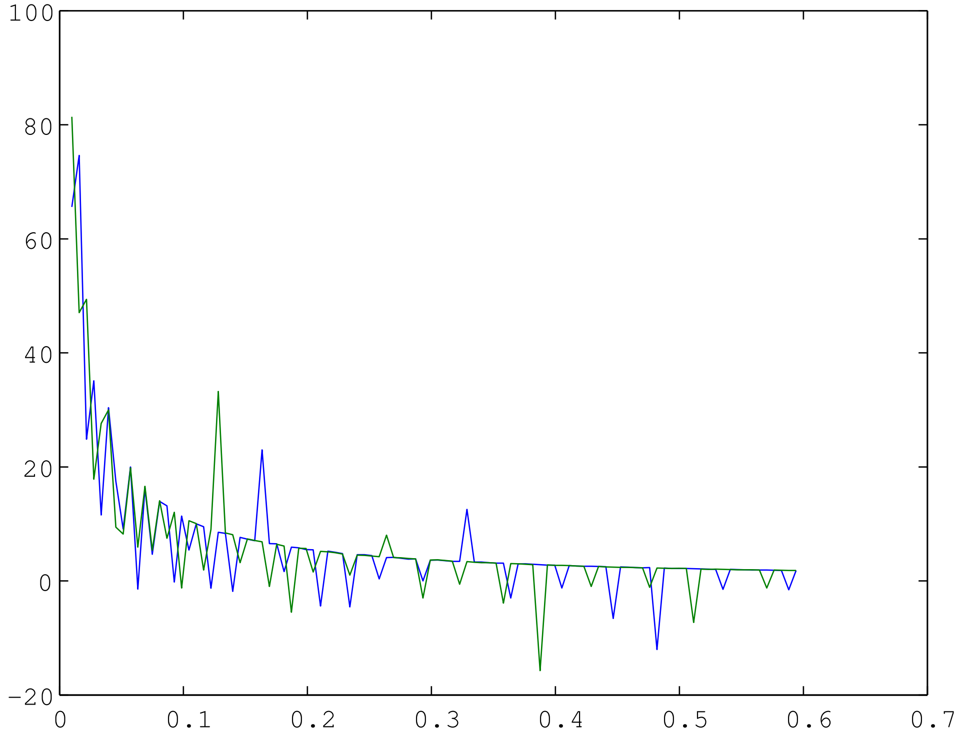

Figure 4: Numerical Derivative of the Value Function when $k = K$

suggesting the presence of more than 1 discontinuity. More to the point, the discontinuity set seems large and dependent on the TFP shock, Z. Thus, it would be difficult to know when we have a model with a well or an ill (as depicted in figure 3) behaved steady state. These jumps are specially relevant near the numerical long run distribution. Below we plot figure 4 for the points in the grid that have positive mass in the long run (i.e. $[Mean - 2 * STD, Mean + 2 * STD]$ for the updating rule and grid size given by $MCes, K_{UB} = 1.5$).

Even though near the mean, 0.3098, the derivative seems continuous, the existence of any unconditional moment depend on a well defined invariant measure (i.e. on a fixed point of the markov kernel P_H). Thus, the jumps to the left and right of the numerical mean are relevant for the existence of a steady state as depiected in figure 3. More importantly, given the finite cardinality of the grid, the algorithm may not capture the discontinuity and display a well behaved histogram.

As discussed in previous sections the observed bias could be generated either by the lack of convergence of the percieved to the actual law of motion (i.e. $H_j \to H_* \nrightarrow h_{*,*}$) or by any difference between the numerical and the actual steady state (i.e. $\mu_i^j \to \mu_i^* \nrightarrow \mu$ for any computed MSS algorithm $i \in 1, 2, 3$ where μ is the fixex point of P_Ψ). As the regards

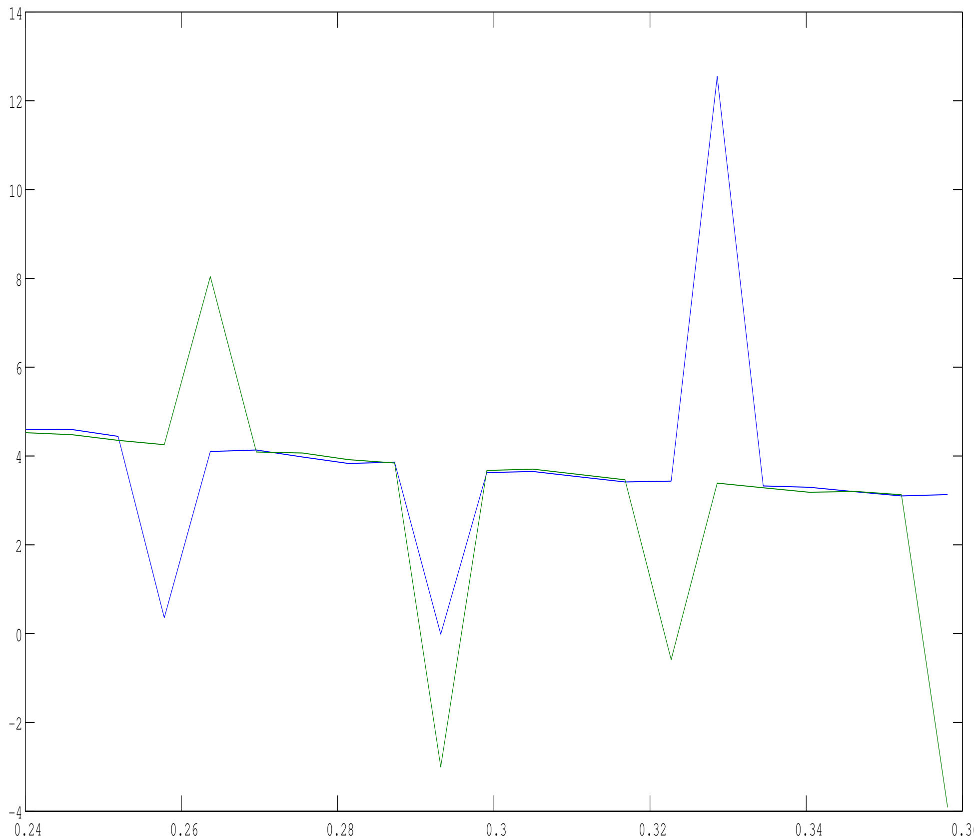

Figure 5: Numerical Derivative of the Value Function in the long run

the former, note that any MSSRE must satisfy equation (14) which in turn insures that any path generated using $h_{*,*}$ along the recursive equilibrium will also be a sequential equilibrium. In other words, any path generated from a MSSRE satisfies equations (7) and (8). The lack of coincidence between the percieved and the actual law of motion will generate a distribution of capital that does not belong to any possible sequential competitive equilibrium, which explains part of the bias as measured in Table 3. Moreover, as a continuous MSSRE may not exist for this model, we cannot insure the existence of a well behaved steady state for this type of equilibria (i.e. μ_{MSSRE} may not exist). If that is the case, any numerical distribution, namely μ^j_{MSSRE}, could be arbitrarily far away from μ as it is not possible to show that $\mu_{MSSRE} \rightarrow \mu$.

The figure in the appendix show, for the sake of completness, the phase diagrams depicted in figures 1 and 2. The numerical histograms used to compute the resuts in table 2 are avilable under request.

We turn to the policy implications of the results in this paper. Note that the mean capital stock for the $MSSCes, K_{UB} = 1.5$ algorithm is well below the ergodic mean. Thus, as τ is decreasing with $\tau \rightarrow 0$ for $K \rightarrow K_{UB}$, using the MSS to predic the long run behavior of this model we may conclude that the observed tax rate is positive. Recent results, see for instance [23], have shown that the optimal tax rate is strictly positive in the long run. So, the policy advice would be to sligthly change the observed tax rate, if any. However, the true distribution, with a support close to the upper bound of J, calls for a increase in the tax effective tax rate.

We are not considering the effects of data on the paramert set. If we want to test this model empirically, the meaningful parameters will be different depending on the selected recursive equilibrium notion. In this case, to assess the effects of the bias, we would have to perform a comparative statics analysis, which is outside the scope of this paper.

5 Conclusions

This paper presents an example of an economy with multiple equilibria and continuous policy functions. This type of equilibrium is useful for accurately assessing the predictions of the model as it allows to generate reliable simulations which can be used to generate counterfactuals which are useful to evaluate alternative economic policies.We present a condition, the monotonicity of the Euler equation, that is associated with exact simulations and provide a description of the reasons behind the lack of accuracy of them.We use the closed form nature of the recursive equilibrium and the induced Feller mechanism to test the accuracy of MSS methods. As the results in this paper does not depend on any numerical procedure, they constitute a unique opportunity to asses the performence of state of the art algoritms.

The paper also connects two branches of the recursive literature: the one concerned with the existence of a steady state (see for instance [20]) and the one concerned with the existence of a recursive representation of the sequential equilibria ([12]). We show that there is no equivalence between the continuity of the equilibrium and its uniqueness, a fact that is useful for simulating the model reliably.

It is clear that the results in this paper have to be generalized. In particular, it is necessary to understand the connection between the number of possible exogenous states and the number of distinct economically meaningful recursive equilibria. That is, as the degree of the polynomial in the equilibrium equation is increasing in the number of exogenous states and each root of the polynomial defines a different mechanism (provided that the root is real and consumption / capital are positive), there is a trade off between a realistic shock process and the predictive performance of the model as more than one possible mechanism generates a less conclusive model. Moreover, the condition required for (1) is sufficiently general to be used in other branches of the literature such as default models.

6 Appendix

6.1 Figures

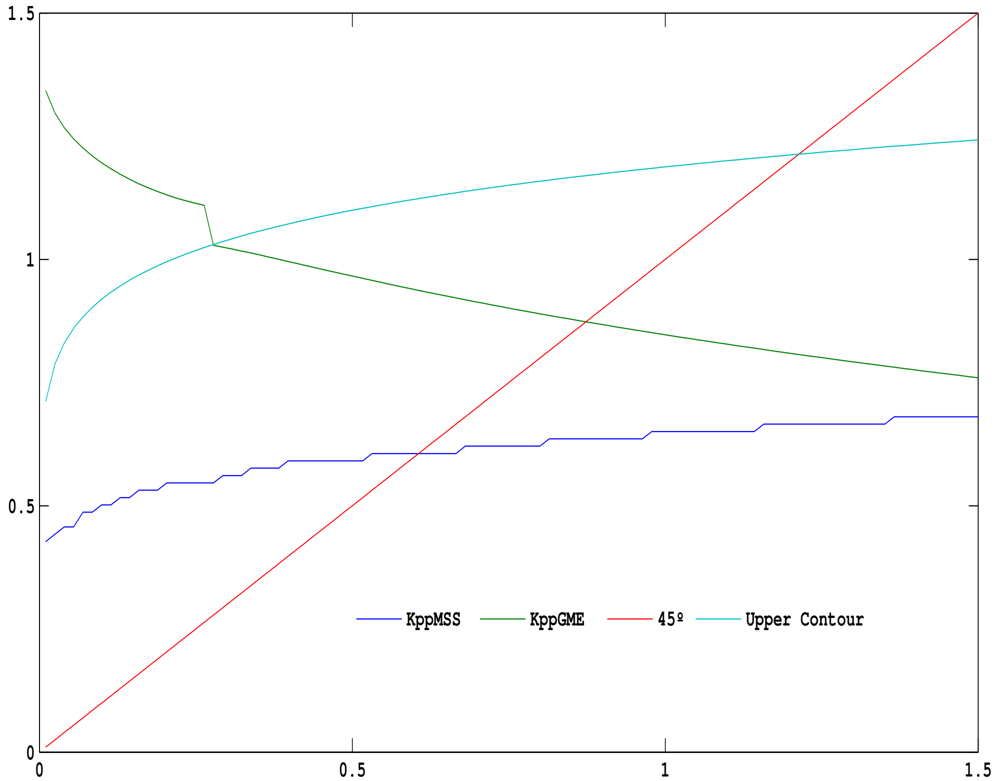

Figure 6: Phase Diagram

The light blue curve is the upper contour for K_{++}, the blue curve is $H(H(K,Z),Z)$ for the MSS and the green line is $H(K,K_{+},Z)$ for the GME, where K_{+} is fixed in the 50th grid point.

6.2 Theorems and useful definitions

A.1 Invariant Measures

Let S be the state space and P a markov operator of the process (A, P). σ_S is the Borel sigma algebra generated by S. P has the Feller property if $P(s, A)$ is continuous (in s) for any $A \subseteq S$. P is tight if S is compact. The operator P maps the space of Borel σ_S -measures $\mathbb{P}$ into itself as follows: $\mu'(A) = \int P(s, A)\mu(ds) \equiv P\mu$.

Theorem A1 (Futia, 1982, page 383, Th. 2.9) If P has the Feller property and is tight, then there is a measure μ such that $\mu = P\mu$

A.2 Supergradients of Concave functions

The following paragraphs are borrowed from [18]. A function $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitzian (LL) if : $|f(x'') - f(x')| \leqslant \lambda |x'' - x'|$, where $\lambda \geqslant 0$ and x'', x' belong to a neighborhood of $x \in R^n$. A concave function is LL. Moreover, the generalized directional derivative (GDD) is given by:

$$f^\circ(x, v) = limsup_{\ x' \to x,\ t \downarrow 0}\ \tfrac{f(x'+tv) - f(x')}{t}$$

When f is LL, the GDD is finite. However, the GDD may be "bizarrely disassociated" from f (see [18] page 5). Thus, we need to connect $f^\circ(x, v)$ with the "classical" directional derivative (DD):

$$f^{'}(x, v) = lim_{t \downarrow 0}\ \tfrac{f(x+tv) - f(x)}{t}$$

We know from [18] (see page 6), that when f is concave, $f^\circ(x, v) = f^{'}(x, v)$ for all x, v. Of course if f is defferentiable $f^{'}(x, v) = f'(x).v$, where f' is the gradient. Moreover, let the superdifferential ∂f be defined as:

$$\partial f = \{p \in \mathbb{R}^n \mid f(x) + p \cdot (y - x) \geqslant f(y),\ \ x, y \in \mathbb{R}^n\}$$

For concave functions at interior points ∂f is non-empty, finite and $p \in \partial f$ satifies $p.v \geqslant f^{'}(x, v)$. Thus, we have a connection between the tangent p of a concave function and it's DD, which is finite at interior points. Finally, if $f : \mathbb{R} \to \mathbb{R}$, the left and right derivative ($f'(x^-), f'(x^+)$ respectively) satisfy $f'(x^-) \geqslant f'(x^+)$ and f' (the derivative) has at most a countable discontinuity set. The left derivative is a minor simplification with respect to $f^{'}(x, v)$:

$$f^{'}(x^-) = lim_{t \downarrow 0}\ \tfrac{f(x-t) - f(x)}{t}$$

A.3 Classical Dynamic Programming

The following paragraphs are borrowed from [22]. Note that equation (13) and the feasibility correspondence Γ define a standard dynamic programming program, as in [22], with states (k, Z) for a given K. In order to prove the strict concavity of $V(k, ., .)$ (Theorem 9.8, page 265) we need the following set of assumptions: i) $k \in X \subset \mathbb{R}^n$, ii) $Z \in \mathbb{Z}$ and $\mathbb{Z}$ is countable, iii) $\Gamma(k, K, Z)$ is continuous in k, iv) Let A be the graph of $\Gamma(k, K, Z)$. Then, $u : A \to \mathbb{R}$ is bounded and continous, v) u is strictly concave for each $Z \in \mathbb{Z}$, vi) $\Gamma(k, K, Z)$ is convex for each $Z \in \mathbb{Z}$. If additionally, we assume that u is differentiable in the interior of A for each $Z \in \mathbb{Z}$, $V(k, ., .)$ is continuously differentiable (Theorem 9.10, page 266).

Unfortunatelly, when we look at Γ when $k = K$, we loose some properties listed above. As τ is decreasing, $(1 - \tau(K))K$ is increasing and, as F is strictly concave in K, $r(K, Z)$ is decreasing. Moeover, given the functional for for F in Table 1, as τ is piecewise linear continuous, (see [19]), $(1-\tau(K))Kr(K)$ is convex, which implies that for some $y \in \Gamma(x, Z)$, $y' \in \Gamma(x', Z)$, we have $\theta y + (1 - \theta)y' \notin \Gamma(\theta x + (1 - \theta)x', Z)$. Thus, we may fail to have a concave and differentiable value functions as property vi) is not satisfied. We need thus to use some properties of LL functions, which are described below.

A.4 Lattice Dynamic Programming and Supermodularity

The following paragraphs are borrowed from [1] and [3]. If equation (13) has interior solutions, $V'_*(k^-, ., .), V'_*(k^+, ., .)$ exist every where for each K, Z, in particular when $k = K$ (see [3], Lemma 3.3). As $u(c) = ln(c)$, we know that solutions will be interior. From Lemma 12 in [1] we know that V_* is LL. The results in A.2 imply in turn that $V^\circ_*(k, ., .)$ is finite, thus, $V'_*(k^-, ., .), V'_*(k^+, ., .)$ are finite.

As $(k, K) \in \left[0, \overline{K}\right] \times \left[0, \overline{K}\right]$ and $\mathbb{Z}$ has finite cardinality and it is bounded, we know that the domain of V_* is a complete lattice (a POSET endowed with the pointwise order such that each pair of elements in it, has a least upper bound $\wedge$ and a greatest lower bound $\vee$ that belong to $\left[0, \overline{K}\right] \times \left[0, \overline{K}\right]$). Then, V_* is supermodular if: $V_*(x \vee y, Z) + V_*(x \wedge y, Z) \geqslant V_*(x, Z) + V_*(y, Z)$. This concept is not really useful. Fortunately, we have an alternative characterization which is called increasing difference (ID). V_* has ID if: $V_*(k, K_1, Z) - V_*(k, K_2, Z)$ is non-decreasing in k for $K_1 \geqslant K_2$. [1] propose a set of sufficient conditions (see assumption E in page 78) in order to insure that V_* has ID which in turn imply that the operator T is convrgent in a very precise sense. We will only mention the assumptions that are not satisfied in the model presented in this paper.

Assumption A.4.1 $u'(c(k, K, Z))(1-\tau(K))r(K)$ is increasing in K and $0 \leqslant c(k, K', Z) - c(k, K, Z) \leqslant F(K', Z) - F(K, Z)$ with $K' \geqslant K$.

If assumption A.4.1 is satisfied, not only V_* has ID (Lemma 12) but also: $T^{\vee j}(F) \to h^\vee_{*,*}$, where $T^{\vee j}(F)$ is the j-th iteration of T starting at F taking the supremun of each maximal element in $ArgmaxV_*$ and $h^\vee_{*,*}$ is the supremun in the set of fixed points of T.

Moreover, $T^{\wedge j}(0) \rightarrow h^{\wedge}_{*,*}$, where the interpretation is analogous. Now we turn to the result in [7], which are generalized in [1].

Assumption A.4.2 $u'(c(k,K,Z))(1-\tau(K))r(K)$ is decreasing in K and $0 \leqslant c(k,K',Z)-c(k,K,Z) \leqslant F(K',Z)-F(K,Z)$ with $K' \geqslant K$.

If assumption A.4.2 is satisfied we can define an operator, based on (9), which insures that there exist a MSSRE and that it can be computed by successive approximations (see [1], theorem 10, page 86). As $(1-\tau(K))r(K)$ is non-monotonic, we can not use any of these results.

A.5 Stationary Markov Equilibria

The results in this section are borrowed from [9]. Let $\Psi : X \rightarrow X$ be the correspondence which defines the GME (see definition 2). Let $C_0 = \left[0, \overline{K}\right] \times \left[0, \overline{K}\right] \times \mathbb{Z}$. Then, we can define a sequence of sets as follows:

$$C_1 = \{x_0 \in X \mid \Psi(x_0) \cap C_0 \neq \varnothing\} \equiv Q(C_0)$$

Let $\{C_i\}_i$ be the sequence of sets generated iteratively using Q. If C_i is non-empty and compact, then $\cap_i C_i = J$ is non-empy and compact and satisfies the self-generation property (i.e. $x \in J$ implies $\Psi(x) \cap J \neq \varnothing$). Intuitively, J is a stationary state space for the markov process generated by P_Ψ. Note that we have modified Q in section 4.2 in order to insure that consumption is positive along the equilibrium path. Thus, we can not use this theorem in order to prove that J is well defined. However, we found a parameter structure and a mechanism which gives a stationary state space numerically.

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